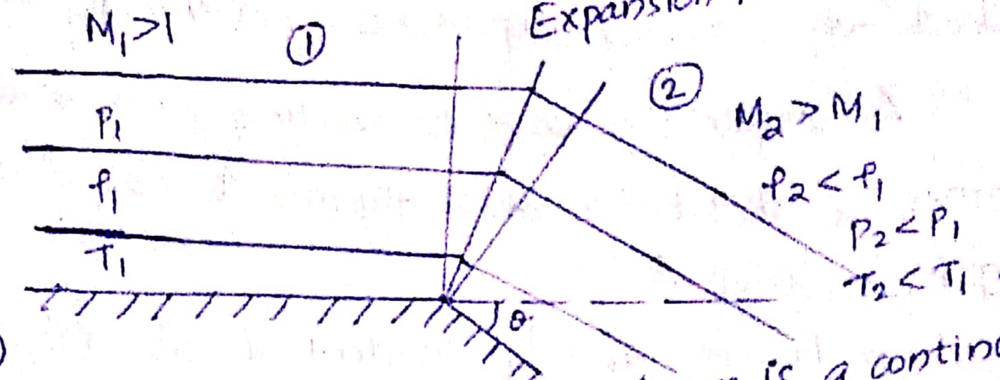
* Prandtl-Meyer Expansion waves:

Oblique shock waves occur when a supersonic flow is turned into itself. In contrast, when a supersonic flow is turned away from itself, an expansion wave is formed. This theory allows us to calculate the changes in flow properties across such expansion waves.



fig(1)

The expansion fan in the above figure is a continuous expansion region which can be visualized as an infinite number of Mach waves, each making the Mach angle μ with the local flow direction.

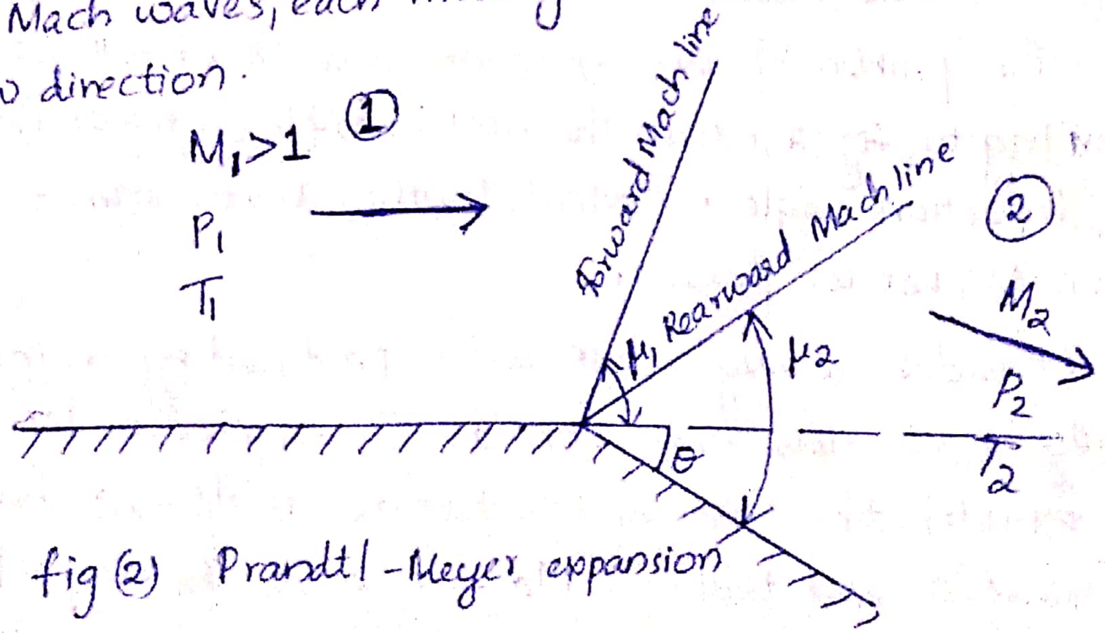


fig (2) Prandtl-Meyer expansion

The expansion fan is bounded upstream by a Mach wave which makes the angle μ_1 w.r.t the upstream flow, where

$\mu_1 = \sin^{-1}\left(\frac{1}{M_1}\right)$.

The expansion fan is bounded downstream by another Mach wave which make the angle μ_2 w.r.t the downstream flow, where $\mu_2 = \sin^{-1}\left(\frac{1}{M_2}\right)$

Since the expansion through the wave takes place across a continuous succession of Mach waves and since $ds = 0$ for each Mach wave, the expansion is isentropic. This is in direct contrast to flow across an oblique shock, which always experiences an entropy increase.

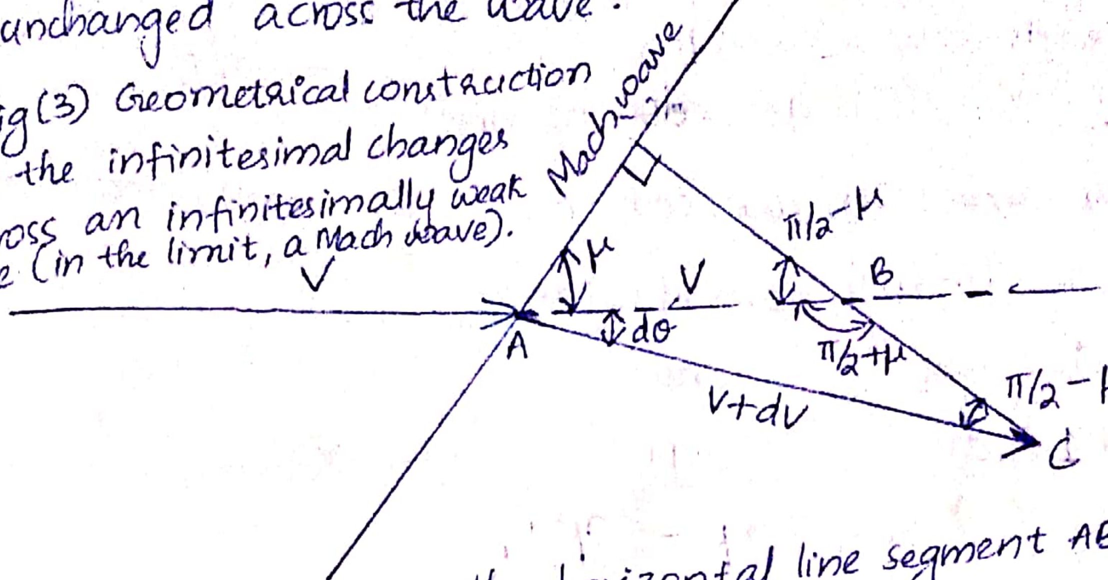
An expansion wave emanating from a sharp convex corner as sketched in above figures is called a <u>centered expansion wave</u>.

Ludwig Prandtl and his student Theodor Meyer first worked out a theory for centered expansion waves in 1907-1908, and hence such waves are commonly denoted as Prandtl-Meyer expansion waves.

The problem of an expansion wave is as follows: referring to fig (2), given the upstream flow (region 1) and the deflection angle θ, calculate the downstream flow (region 2). Let us proceed.

Consider a very weak wave produced by an infinitesimally small flow deflection $d\theta$ as sketched in fig (3). We consider the limit of this picture as $d\theta \to 0$; hence, the wave is essentially a Mach wave at the angle 'μ' to the upstream flow. The velocity ahead of the wave is 'V'. As the flow is deflected downward through the angle $d\theta$, the velocity is increased by the infinitesimal amount dV, and hence the flow velocity behind the wave is $V+dV$ inclined at the angle $d\theta$. Recall from the treatment of the momentum eqn that any change in velocity across a wave takes place normal to the wave; the tangential component is unchanged across the wave.

fig (3) Geometrical construction for the infinitesimal changes across an infinitesimally weak wave (in the limit, a Mach wave).



In the above figure, the horizontal line segment AB with length V is drawn behind the wave.
Also, the line segment AC is drawn to represent the new velocity V+dV behind the wave.
Then line BC is normal to the wave because it represents the line along which the change in velocity occurs.
Examining the geometry in fig (3), from the law of sines applied to Δle ABC, we get:

$$\frac{\sin d\theta}{BC} = \frac{\sin\left(\frac{\pi}{2}+\mu\right)}{V+dV} = \frac{\sin\left(\frac{\pi}{2}-\mu-d\theta\right)}{V} \qquad \left\{ \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \right\}$$

From the above eqn:

$$\frac{V+dV}{V} = \frac{\sin(\pi/2+\mu)}{\sin(\pi/2-\mu-d\theta)} \longrightarrow (1)$$

However, from trigonometric identities:

$$\sin(\pi/2+\mu) = \sin(\pi/2-\mu) = \cos\mu \longrightarrow (2)$$

$$\sin(\pi/2-\mu-d\theta) = \cos(\mu+d\theta) = \cos\mu\cos d\theta - \sin\mu\sin d\theta \longrightarrow (3)$$

Substituting eqns (2) & (3) in (1), we get:

$$\frac{V+dV}{V} = \frac{\cos\mu}{\cos\mu\cos d\theta - \sin\mu\sin d\theta}$$

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$$1+\frac{dv}{v} = \frac{\cos\mu}{\cos\mu\cos d\theta - \sin\mu\sin d\theta} \longrightarrow (4)$$

For small $d\theta$, we can make the small-angle assumptions $\sin d\theta \simeq d\theta$ and $\cos d\theta \cong 1$.

then eqn (4) becomes:

$$1+\frac{dv}{v} = \frac{\cos\mu}{\cos\mu - d\theta\sin\mu} \longrightarrow (5)$$

Now take $\cos\mu$ common in above eqn:

$$\therefore 1+\frac{dv}{v} = \frac{1}{1-d\theta\tan\mu} \longrightarrow (6)$$

Note that the function $\frac{1}{1-x}$ can be expanded in a power series (for $x<1$) as:

$$\frac{1}{1-x} = 1+x+x^2+x^3+\text{-----}$$

$$\therefore 1+\frac{dv}{v} = 1+d\theta\tan\mu + \text{----}$$

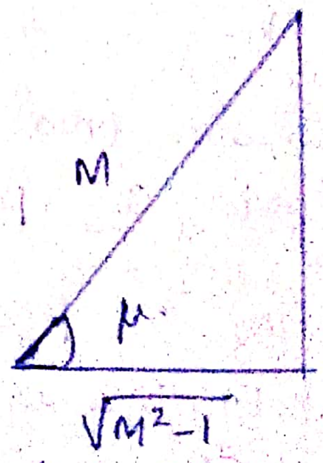
By ignoring second order and higher terms, we get:

$$1+\frac{dv}{v} = 1+d\theta\tan\mu \longrightarrow (7)$$

$$d\theta = \frac{dv/v}{\tan\mu} \longrightarrow (8)$$

we know $\mu = \sin^{-1}\left(\frac{1}{M}\right)$.

from the Δle, $\tan\mu = \frac{1}{\sqrt{M^2-1}}$.



$$\therefore \boxed{d\theta = \sqrt{M^2-1}\,\frac{dv}{v}} \longrightarrow (9)$$

Eqn (9) relates the infinitesimall change in velocity, dv, to the infinitesimal deflection $d\theta$ across a wave of vanishing strength.

In the precise limit of a mach wave, of course dV and hence $d\theta$ are zero. Eqn ⑨ is an approximate equation for a finite $d\theta$, but it becomes a true equality as $d\theta \to 0$. Since the expansion fan illustrated in figures is a region of an infinite number of Mach waves, Eqn ⑨ is a differential equation which precisely describes the flow inside the expansion wave.

Let us integrate eqn ⑨ from region 1, where the deflection angle is zero and mach number is M_1, to region 2, where the deflection angle is θ and the Mach number is M_2.

$$\therefore \int_0^\theta d\theta = \theta = \int_{M_1}^{M_2} \sqrt{M^2-1}\,\frac{dV}{V} \longrightarrow (10)$$

To carry out the integral on the R.H.S, $\frac{dV}{V}$ must be obtained in terms of 'M' as follows:

we know $M = \frac{V}{a} \Rightarrow V = Ma$

$\therefore \ln V = \ln M + \ln a.$ $\quad [a = \sqrt{\gamma RT}]$

By taking differentiation, we get:

$$\frac{dV}{V} = \frac{dM}{M} + \frac{1}{2}\frac{dT}{T} \longrightarrow (11)$$

we know, $\frac{T}{T_0} = \left[1 + \frac{\gamma-1}{2}M^2\right]^{-1}$

By logarithmic differentiation, we get:

$$\frac{dT}{T} = -\frac{(\gamma-1)M^2}{1+\frac{\gamma-1}{2}M^2}\,\frac{dM}{M}$$

Substituting in eqn (11):

$$\therefore \frac{dV}{V} = \frac{dM}{M} + \frac{1}{2}\left[\frac{-(\gamma-1)M^2}{1+\frac{\gamma-1}{2}M^2}\right]\frac{dM}{M}$$

$$\therefore \boxed{\frac{dV}{V} = \frac{dM}{M}\left[\frac{1}{1+\frac{\gamma-1}{2}M^2}\right]} \longrightarrow (12)$$

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Substitute eqn (12) in eqn (10):

$$\boxed{\theta = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}} \longrightarrow (13)$$

In eqn (13)

$$\boxed{\nu(M) = \int \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}} \longrightarrow (14)$$

This integral is called Prandtl-Meyer function and is denoted by ν.

After integrating, eqn (14) becomes:

$$\boxed{\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1}}$$

For convenience, the integral constant is considered as zero.

$\Rightarrow$ when $M=1$, $\nu(M)=0$.

• We can rewrite eqn (13) combined with eqn (14) as:

$$\boxed{\theta = \nu(M_2) - \nu(M_1)}$$

→ The Prandtl Meyer function ν is very important; it is the key to the calculation of changes across an expansion wave.

→ The expansion wave is isentropic, hence P_0 and T_0 are constant through the wave. That is $T_{0,2} = T_{0,1}$ and $P_{0,2} = P_{0,1}$.

We know that:

$$\frac{T_0}{T} = 1 + \frac{\gamma-1}{2}M^2.$$

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So, $\frac{T_{02}}{T_2} = 1 + \frac{\gamma-1}{2} M_2^2$; $\frac{T_{01}}{T_1} = 1 + \frac{\gamma-1}{2} M_1^2$

$$\therefore \frac{\frac{T_{01}}{T_1}}{\frac{T_{02}}{T_2}} = \frac{1 + \frac{\gamma-1}{2} M_1^2}{1 + \frac{\gamma-1}{2} M_2^2}$$; where $T_{01} = T_{02}$

$$\therefore \boxed{\frac{T_2}{T_1} = \frac{1 + \frac{\gamma-1}{2} M_1^2}{1 + \frac{\gamma-1}{2} M_2^2}}$$

Illy, $$\boxed{\frac{P_2}{P_1} = \frac{P_2/P_{02}}{P_1/P_{01}} = \left[\frac{1 + \frac{\gamma-1}{2} M_1^2}{1 + \frac{\gamma-1}{2} M_2^2}\right]^{\frac{\gamma}{\gamma-1}}}$$; where $P_{02} = P_{01}$

Since we know both M_1 and M_2 as well as T_1 and P_1, both equations allow the calculation of T_2 and P_2 downstream of the expansion wave.

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* Fanno flow: (Flow in a constant area duct with friction)

Consider the one-dimensional steady flow of a perfect gas, with constant specific heats through a constant area duct with friction.

Also let there be neither external heat exchange nor external shaft work and assume the differences in elevation produce negligible changes as compared ~~to~~ with frictional effects.

The flow with above-mentioned conditions, namely adiabatic flow with no external work is called Fanno line flow.

Let the wall friction (due to viscosity) be the chief factor bringing about changes in fluid properties, for the adiabatic compressible flow through ducts of constant area under consideration.

→ The energy equation of steady flow under the above assumptions is:

$$h + \frac{V^2}{2} = h_0 \quad \text{—— (1)}$$

where h & V are respectively the corresponding values of the enthalpy and velocity at an arbitrary section of the duct and h_0 (the stagnation enthalpy) has a constant value for all sections of the duct.

→ By equation of continuity:

$$\dot{m} = \rho A V$$

$$\frac{\dot{m}}{A} = \rho V = G \quad \text{—— (2)}$$

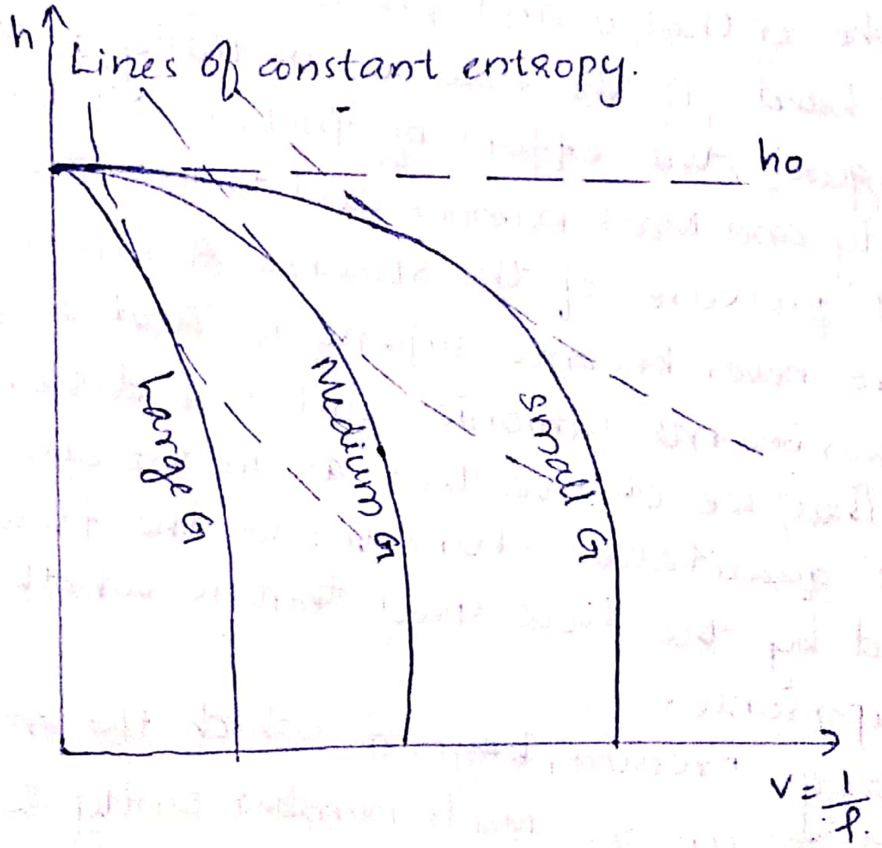
where ρ is the density at the section where V and h are measured and G is called the mass velocity, which has a constant value for all sections of the duct.

Combining eqns ① & ②, we get the eqn of Fanno line in terms of the enthalpy and density as:

$$\boxed{h = h_0 - \frac{G^2}{2\rho^2}} \longrightarrow ③$$

Because h_0 and G are constants for a given flow. Eqn ③ defines a relation b/w the local density and the local enthalpy. This relation defines families of curves (the particular curve depending on the choice of the parameters G and h_0) in the plane of any two thermodynamic variables. In below figure, this relation is shown graphically in the h-v plane, for a single value of h_0 and for several values of G. Such curves, in general are called Fanno lines.

* Fanno lines on a h-v plane.



* The Fanno line:

For a pure substance:

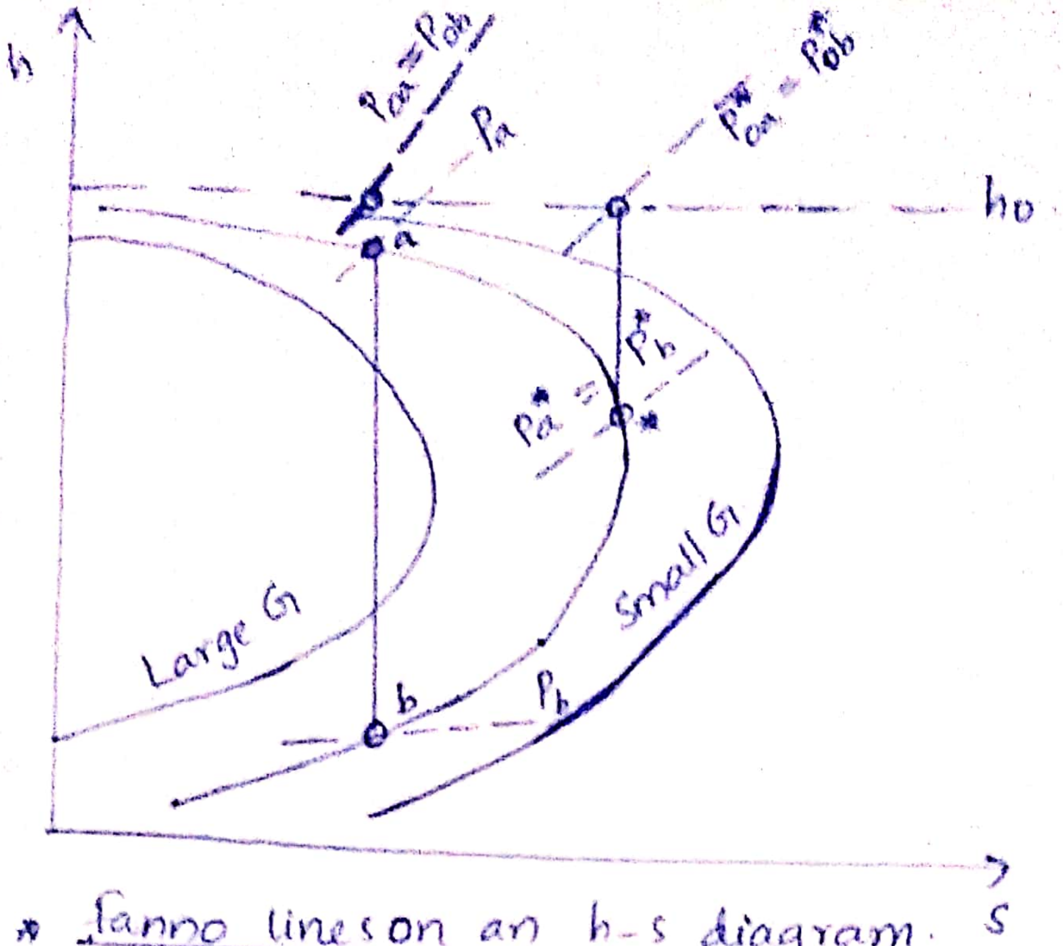
$$s = s(h, \rho).$$

That is, the entropy is determined by the enthalpy and the density. The curves of the above figure may thus be transferred to the enthalpy-entropy diagram, giving the familiar Fanno curves of the below figure. For all substances so far investigated, the Fanno curves have the same general shape. The three curves shown in below figure have the same stagnation enthalpy but different mass flow rates per unit area.

We know by the IInd law of thermodynamics that for an adiabatic flow the entropy may increase but cannot decrease. (ie, $\Delta s \geq 0$). Thus in the below figure, the path of states along any one of the Fanno curves must be towards the right. Therefore, if the flow at some point in the duct is <u>subsonic</u> (point a in figure), the effect of friction will be to increase the velocity and Mach number and to decrease the enthalpy and pressure of the stream. On the other hand, if the flow is initially <u>supersonic</u> (point b in figure), the effect of friction will be to decrease the velocity and Mach number and to increase the enthalpy and pressure of the stream. A subsonic flow may therefore never become supersonic and a supersonic flow may never become subsonic, unless a discontinuity is present. Thus, we observe that, as in the case of isentropic flow, the qualitative character of the flow is markedly influenced by the flow speed, that is whether it is subsonic or supersonic.

The limiting pressure, beyond which the entropy would decrease, occurs at Mach number unity & is denoted by p^*. The asterisk here denotes the state where $M=1$, for the particular case of adiabatic flow through ducts of constant area.

From the below figure, it can be seen that the isentropic stagnation pressure is reduced as a result of friction, irrespective of whether the flow is subsonic or supersonic.

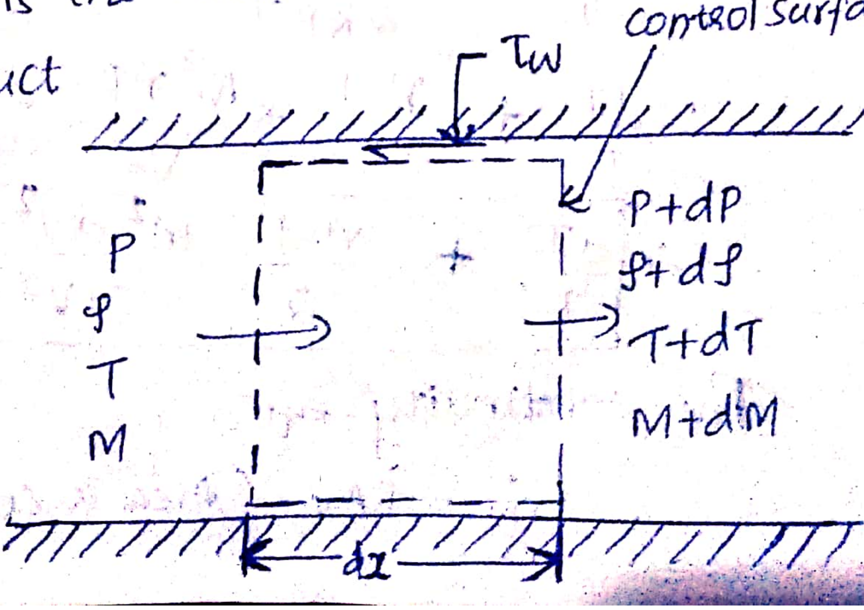


* Fanno lines on an h-s diagram.

* Adiabatic, Constant - Area flow of a Perfect Gas:

In this section, the fluid is assumed to be perfect.
The aim here is to express the variations in flow characteristics along the length of a duct of constant area in analytical form. This requires the introduction of the momentum equation, with a term accounting for the frictional forces acting on the control volume, since the rate of change of flow properties depends on the amount of friction.

Select an infinitesimall control volume as shown in figure.
In the figure, τ_w is the shear stress due to friction acting on the wall of the duct



* Control surface for analysis of adiabatic, constant-area flow.

For a perfect gas, $P = \rho R T$

By taking logarithmic differentiation, we get:

$$\boxed{\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}} \longrightarrow (4)$$

we know $M = \frac{V}{a} \Rightarrow M = \frac{V}{\sqrt{\gamma R T}} \Rightarrow M^2 = \frac{V^2}{\gamma R T}$

By taking logarithmic differentiation, we get:

$$\boxed{\frac{dM^2}{M^2} = \frac{dV^2}{V^2} - \frac{dT}{T}} \longrightarrow (5)$$

Energy equation for a perfect gas is given by:

$$H + \frac{1}{2} m v^2 = \text{constant} \qquad \left[\because \frac{H}{m} = h\right]$$

$$h + \frac{1}{2} V^2 = \text{constant}$$

we know $h = C_p T$

$$\therefore C_p T + \frac{1}{2} V^2 = \text{constant}$$

Now, Differentiate the equation:

$$\therefore C_p dT + \frac{1}{2} dV^2 = 0$$

Divide by $C_p T$.

$$\therefore \frac{dT}{T} + \frac{1}{2 C_p T} dV^2 = 0. \qquad \left[\because C_p = \frac{R\gamma}{\gamma - 1}\right]$$

$$\frac{dT}{T} + \frac{\gamma - 1}{2 R \gamma T} dV^2 = 0.$$

$$\therefore \frac{dT}{T} + \frac{\gamma - 1}{2 R \cancel{\gamma}} \times \frac{M^2 \cancel{\gamma R}}{V^2} dV^2 = 0. \qquad \left[\because \frac{1}{T} = \frac{M^2 \gamma R}{V^2}\right]$$

$$\therefore \boxed{\frac{dT}{T} + \frac{\gamma - 1}{2} M^2 \frac{dV^2}{V^2} = 0} \longrightarrow (6)$$

The continuity eqn:

$\dot{m} = \rho A V$ (Area is constant).

$$\therefore \boxed{\frac{\dot{m}}{A} = \rho V = G} \quad ; G \text{ is constant}.$$

$\therefore \rho V$ is also constant ; ρV = constant

By logarithmic differentiation:

$$\boxed{\frac{d\rho}{\rho} + \frac{1}{2}\frac{dV^2}{V^2} = 0} \longrightarrow (7)$$

→ Total force :

$F = ma$ (According to Newton's IInd law).

$F = PA + \tau A$

where $\tau = \tau_w$

$ma = -PA - \tau_w \times A_w$ (Due to pressure and shear stress).

$m \times \frac{V}{t} = -PA - \tau_w \times A_w$

$\dot{m} V = -PA - \tau_w \times A_w$.

Differentiating:

$$\boxed{\dot{m} dV = -A dP - \tau_w \times dA_w} \longrightarrow (8)$$

↓ A constant ↓ τ is constant

Where, A is the cross sectional area of duct and dA_w is the wetted wall surface area of the duct over which the shear stress τ_w acts.

→ Friction Coefficient (Drag Coefficient):

The Coefficient of friction or the Coefficient of drag, as it is generally referred to for flow in ducts, is defined as:

$$f = \frac{\text{wall shear stress}}{\text{Dynamic pressure head of the stream.}}$$

$$\boxed{f = \frac{\tau_w}{\frac{1}{2}\rho V^2}} \longrightarrow (9)$$

It is common practice in such analysis to use a parameter called hydraulic diameter D.

* Hydraulic diameter (D):

It is defined as:

$$D = \frac{4\,(\text{cross-sectional area})}{\text{Wetted perimeter}}$$

$$\boxed{D = \frac{4A}{\left(\frac{dA_w}{dx}\right)} = \frac{4A}{dA_w}\,dx} \longrightarrow (10)$$

The advantage of using hydraulic diameter is that the equation in terms of hydraulic diameter are valid even for ducts with non-circular cross-section.

→ Sub Eqns (9) & (10) in eqn (8):

$$-A\,dP - \frac{1}{2}\rho V^2 f \frac{4A}{D}\,dx = \dot{m}\,dV$$

Divide by A:

$$-dP - \frac{1}{2}\rho V^2 f \frac{4}{D}\,dx = \frac{\dot{m}}{A}\,dV.$$

$$= \frac{\rho A V}{A}\,dV \times \frac{V}{V}$$ [Multiply & divide by V to obtain a term $\frac{dV}{V}$]

$$= \rho V^2 \frac{dV}{V}$$

$$\therefore \rho V^2 \frac{dV}{V} + dP + \frac{1}{2}\rho V^2 f \frac{4}{D}\,dx = 0.$$

Divide by P:

$$\boxed{\frac{\rho V^2}{P}\frac{dV}{V} + \frac{dP}{P} + \frac{1}{2}\frac{\rho V^2}{P} f \frac{4}{D}\,dx = 0} \longrightarrow (11)$$

where $\rho V^2 = \rho \times \frac{M^2 \gamma P}{\rho}$ $\left[\text{from } M^2 = \frac{V^2}{\gamma RT};\ P = \rho RT \Rightarrow \frac{P}{\rho} = RT\right]$

$$\boxed{\rho V^2 = \gamma M^2 P}$$

Substitute in Eqn (11):

$$\frac{\gamma R M^2}{R}\frac{dV}{V}+\frac{dP}{P}+\frac{1}{2}\frac{\gamma M^2 R}{R} f \frac{4}{D}dx=0.$$

$$\therefore \boxed{\gamma M^2\frac{dV}{V}+\frac{dP}{P}+\frac{1}{2}\gamma M^2 f\frac{4}{D}dx=0} \longrightarrow (12)$$

$\frac{dV}{V}$ can also be written as $\frac{1}{2}\frac{dV^2}{V^2}$

$$\boxed{\frac{\gamma M^2}{2}\frac{dV^2}{V^2}+4f\frac{\gamma M^2}{2}\frac{dx}{D}+\frac{dP}{P}=0} \longrightarrow (13)$$

we know that:

$$\frac{P_0}{P}=\left[1+\frac{\gamma-1}{2}M^2\right]^{\frac{\gamma}{\gamma-1}}$$

$\therefore$ Isentropic stagnation pressure, $P_0=P\left[1+\frac{\gamma-1}{2}M^2\right]^{\frac{\gamma}{\gamma-1}}$

By logarithmic differentiation:

$$\boxed{\frac{dP_0}{P_0}=\frac{dP}{P}+\left[\frac{\gamma M^2/2}{1+\frac{\gamma-1}{2}M^2}\right]\frac{dM^2}{M^2}} \longrightarrow (14)$$

$\Rightarrow$ Impulse Function F is:

$$\boxed{F=PA+ma} \longrightarrow (15)$$

$$\left[\because ma=\frac{mV}{t}=\dot{m}V=\rho AV^2\right].$$

$$\boxed{F=PA+\rho AV^2} \longrightarrow (16)$$

we know that:

$$\rho V^2=\gamma P M^2.$$

Sub in eqn (16):

$$\therefore F=PA+\gamma P M^2 A.$$

$$\therefore \boxed{F=PA\left[1+\gamma M^2\right]} \longrightarrow (17)$$

$[\because A \text{ is constant}]$

By logarithmic differentiation:

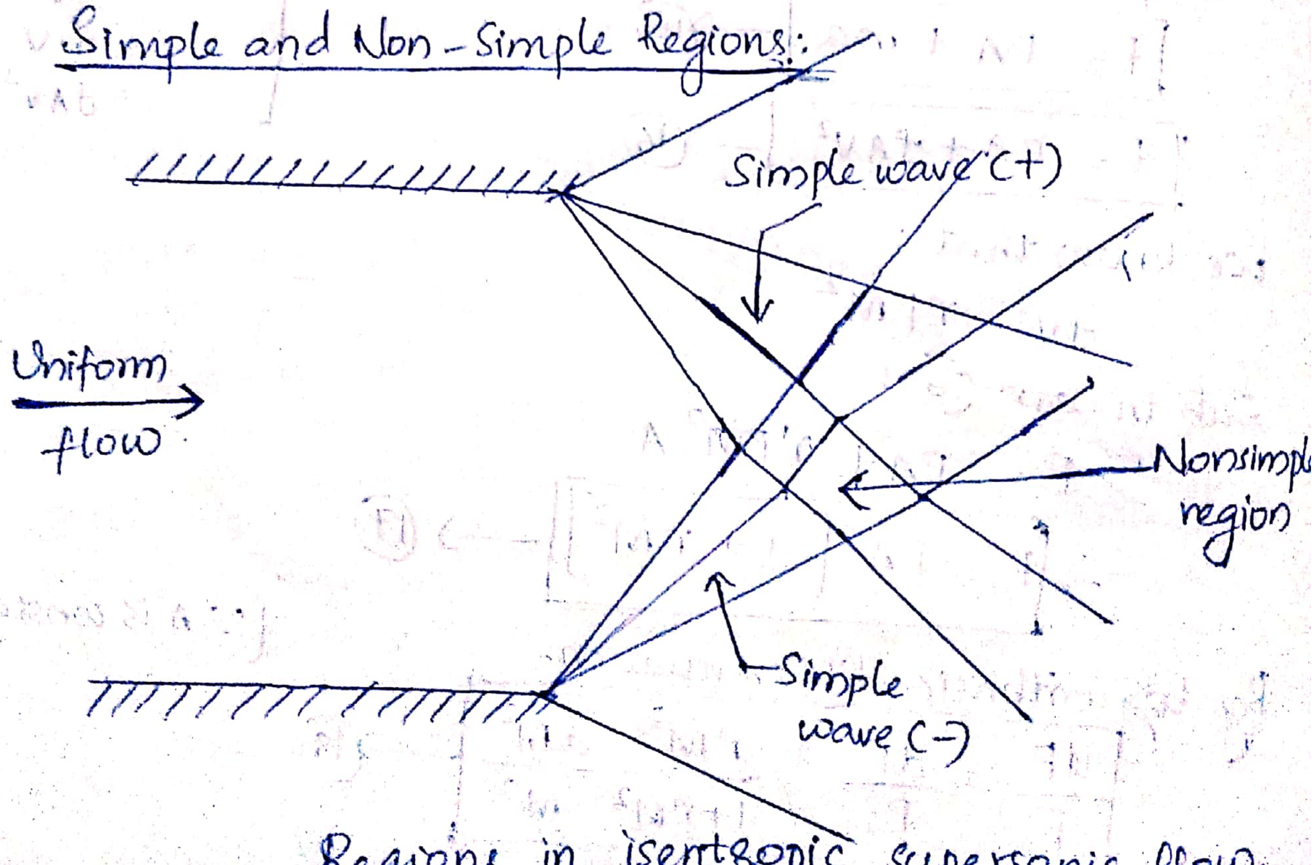
$$\boxed{\frac{dF}{F}=\frac{dP}{P}+\frac{\gamma M^2}{1+\gamma M^2}\frac{dM^2}{M^2}.} \longrightarrow (18)$$

* Instead of continuous change, if the compression is caused by only one kink at the wall of about 10°, the compression cannot be isentropic and the turning of the flow will take place through a shock wave as shown in figure. Only when the compression is continuous, can the process treated as isentropic. However, for small values of θ, the compression through a single compression wave, such as shown in figure, the process can still be treated as isentropic and reasonably accurate results can be obtained. Flow turning angle $\theta \le 5°$ may be taken as the limit for considering the compression to be isentropic.

From the above discussion it is clear that the relation b/w the Prandtl-Meyer function and the flow turning angle may be expressed as:

$$\boxed{\begin{aligned} \nu_n &= \nu_{n-1} + |\theta_n - \theta_{n-1}| \quad \text{(expansion)} \\ \nu_n &= \nu_{n-1} - |\theta_n - \theta_{n-1}| \quad \text{(compression)} \end{aligned}}$$

* <u>Simple and Non-Simple Regions:</u>



Regions in isentropic supersonic flow.

• The waves causing isentropic expansion and compression ~~dis~~ are called Simple waves.

• A simple wave is a straight Mach line, with constant flow conditions, and is governed by the simple relations:

$$V_n = V_{n-1} + |\theta_n - \theta_{n-1}| \quad (\text{expansion})$$

$$V_n = V_{n-1} - |\theta_n - \theta_{n-1}| \quad (\text{compression})$$

between the flow direction and the Prandtl-Meyer function.

• A supersonic flow field with simple and nonsimple regions is shown in the figure.

• A supersonic expansion or compression zone with Mach lines which are straight is called a simple region.

• Equations which govern the simple region are not applicable to non-simple regions.

• The Mach lines which are straight in the simple region become curved in the non simple region, after intersecting with other Mach lines.

• However, the wave segments b/w adjacent cross-over points may be treated as linear, without introducing significant error to the calculated results.

* Derivation Continuation:

from eqn (4) there are 3 differential variables $\rightarrow \frac{dP}{P}, \frac{d\rho}{\rho}, \frac{dT}{T}$

from eqn (5) $\rightarrow \frac{dM^2}{M^2}, \frac{dV^2}{V^2}$

from eqn (14) $\rightarrow \frac{dP_0}{P_0}$

from eqn (12) $\rightarrow 4f\frac{dx}{D}$

From eqn (18) $\rightarrow \frac{dF}{F}$

we have 7 equations and 8 variables. Therefore we cannot solve it.

→ The physical phenomenon causing changes in flow properties is the viscous friction.

→ Choose the variable $4f\frac{dx}{D}$, then solve the 7 eqn as simultaneous eqn for remaining 7 variables.

We obtain

$$\boxed{\frac{dM^2}{M^2} = \frac{\gamma M^2\left[1+\frac{\gamma-1}{2}M^2\right]}{1-M^2} 4f\frac{dx}{D}} \rightarrow (19)$$

$$\boxed{\frac{dP}{P} = -\frac{\gamma M^2\left[1+(\gamma-1)M^2\right]}{2\left[1-M^2\right]} 4f\frac{dx}{D}} \rightarrow (20)$$

$$\boxed{\frac{dV}{V} = \frac{\gamma M^2}{2\left[1-M^2\right]} 4f\frac{dx}{D}} \rightarrow (21)$$

$$\frac{dT}{T} = \frac{1}{2}\frac{da}{a}$$

$$\boxed{\frac{dT}{T} = \frac{-\gamma(\gamma-1)M^4}{2\left[1-M^2\right]} 4f\frac{dx}{D}} \rightarrow (22)$$

$$\boxed{\frac{d\rho}{\rho} = \frac{-\gamma M^2}{2\left[1-M^2\right]} 4f\frac{dx}{D}} \rightarrow (23)$$

$$\boxed{\frac{dP_0}{P_0} = -\frac{\gamma M^2}{2} 4f\frac{dx}{D}} \rightarrow (24)$$

$$\boxed{\frac{dF}{F} = \frac{-\gamma M^2}{2\left[1+\gamma M^2\right]} 4f\frac{dx}{D}} \rightarrow (25) \quad (F = \text{Impulse Function})$$

⇒ For an adiabatic flow, the stagnation temperature is the only invariant.

The entropy change can be expressed as:

$$\boxed{\frac{ds}{C_p} = -\frac{\gamma-1}{\gamma}\frac{dP_0}{P_0}} \rightarrow (26) \quad \left[\begin{array}{l} dQ = Tds \\ dQ = dH - vdp \end{array}\right]$$

Sub eq (24) in eqn (26):

$$\therefore \frac{ds}{C_p} = -\frac{\gamma-1}{\gamma}\left[-\frac{\gamma M^2}{2} 4f\frac{dx}{D}\right]$$

$$\therefore \boxed{\frac{ds}{C_p} = \frac{(\gamma-1)}{2} M^2\, 4f\frac{dx}{D}} \rightarrow (27)$$

⇒ According to the 2nd law of thermodynamics the entropy should not be decreased in an adiabatic flow process.

⇒ So the friction coefficient f must always be a positive quantity and dx also positive in the direction of flow.

⇒ Since f must always be +ve, the shear stress must always act in a direction opposite to the flow.

In the case of subsonic inlet flow, the effect of friction on the downstream flow is such that:

- Pressure p decreases
- Mach number M increases
- Velocity V increases
- Temperature T decreases
- Density ρ decreases
- Stagnation pressure P_0 decreases
- Impulse function F decreases.

For the supersonic inlet flow, the effect of friction on the downstream flow is such that:

- P increases
- M decreases
- V decreases
- T increases
- ρ increases
- P_0 decreases
- F decreases

From the above summary, we may observe that the friction has the net effect of accelerating a subsonic stream and causes a rise in static pressure at supersonic speeds.

Fanno flow is also known as simple friction flow.

Working Relations:

Let the Mach number be the independent variable for this purpose. Then eqn (A) may be rearranged to give:

$$4f\frac{dx}{D} = \frac{1-M^2}{\gamma M^4\left(1+\frac{\gamma-1}{2}M^2\right)}\,dM^2 \longrightarrow (28)$$

Integrate the eqn:

$$\int_0^{L_{max}} 4f\frac{dx}{D} = \int_{M^2}^{1} \frac{1-M^2}{\gamma M^4\left[1+\frac{\gamma-1}{2}M^2\right]}\,dM^2$$

when $M = M^2$; $x = 0$.

$M = 1$; $x = L_{max}$

After integration & Let $\boxed{\frac{1}{L_{max}}\int_0^{L_{max}} f\,dx = \bar{f}}$; where

$\bar{f}$ is the mean friction coefficient w.r.t duct length.

∴ we get:

$$\boxed{4\bar{f}\,\frac{L_{max}}{D} = \frac{1-M^2}{\gamma M^2} + \frac{\gamma+1}{2\gamma}\ln\left[\frac{(\gamma+1)M^2}{2\left(1+\frac{\gamma-1}{2}M^2\right)}\right]} \longrightarrow (29)$$

Eqn (29) gives the maximum value of $4\bar{f}(L/D)$ corresponding to any initial Mach number M.

Because $4\bar{f}(L_{max}/D)$ is a function only of M, the duct length required for the flow to pass from a given initial Mach number M_1 to a given final Mach number M_2 is obtained from the expression:

$$\boxed{4\bar{f}\left(\frac{L}{D}\right) = \left(4\bar{f}\frac{L_{max}}{D}\right)_{M_1} - \left(4\bar{f}\frac{L_{max}}{D}\right)_{M_2}} \longrightarrow (30)$$

IIIly local flow properties can be found in terms of local Mach number. Indicating the properties at M=1 as superscripted with asterisk and integrating b/w the duct sections with Mach number M and 1, we obtain from eqns (20) to (27)

$$\frac{p}{p^*} = \frac{1}{M}\left[\frac{\gamma+1}{2\left(1+\frac{\gamma-1}{2}M^2\right)}\right]^{1/2} \longrightarrow (31)$$

$$\frac{V}{V^*} = M\left[\frac{\gamma+1}{2\left(1+\frac{\gamma-1}{2}M^2\right)}\right]^{1/2} \longrightarrow (32)$$

$$\frac{T}{T^*} = \frac{a^2}{a^{*2}} = \frac{\gamma+1}{2\left(1+\frac{\gamma-1}{2}M^2\right)} \longrightarrow (33)$$

$$\frac{\rho}{\rho^*} = \frac{V^*}{V} = \frac{1}{M}\left[\frac{2\left(1+\frac{\gamma-1}{2}M^2\right)}{\gamma+1}\right]^{1/2} \longrightarrow (34)$$

$$\frac{P_0}{P_0^*} = \frac{1}{M}\left[\frac{2\left(1+\frac{\gamma-1}{2}M^2\right)}{\gamma+1}\right]^{\frac{\gamma+1}{2(\gamma-1)}} \longrightarrow (35)$$

$$\frac{F}{F^*} = \frac{1+\gamma M^2}{M\left[2(\gamma+1)\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{1/2}} \longrightarrow (36)$$

$$\frac{S-S^*}{C_p} = \ln M^2\left[\frac{\gamma+1}{2M^2\left(1+\frac{\gamma-1}{2}M^2\right)}\right]^{\frac{\gamma+1}{2\gamma}} \longrightarrow (37)$$

IIIly $$\frac{\rho_2}{\rho_1} = \frac{\left(\frac{\rho}{\rho^*}\right)_{M_2}}{\left(\frac{\rho}{\rho^*}\right)_{M_1}} \longrightarrow (38)$$

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We know that the quantities marked with an asterisk in these equations are constant for a given adiabatic constant-area flow. Therefore, they may be regarded as convenient reference values for normalizing the equations. To find the change in a flow property, say the density, b/w sections of the duct where the Mach numbers are M_1 and M_2, we set eqn (38).

where $\left(\frac{\rho}{\rho^*}\right)_{M_1}$ is the value on the R.H.S of eqn (34) corresponding to M_1 and so on.

* Rayleigh flow (simple heating flow)

It is also known as simple heating flow. It is a constant area flow without frictional effects. It includes changes in stagnation enthalpy & stagnation temperature.

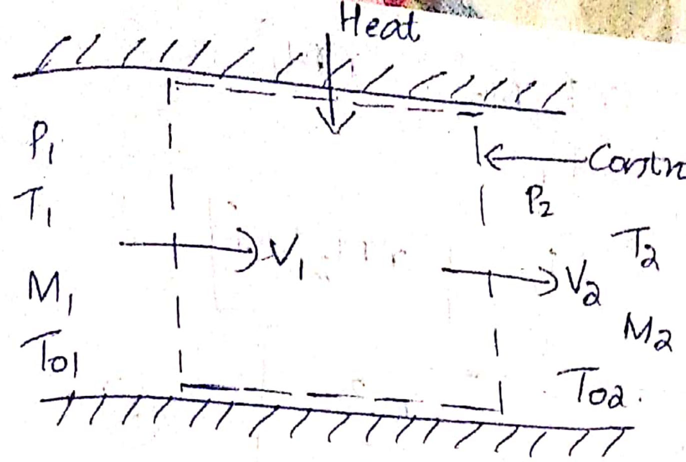
The flow involving change in stagnation temperature or the stagnation enthalpy of a gas stream which flows at constant area and without frictional effects are known as Rayleigh flow.

* Derivation:

→ Consider the flow of a perfect gas through a constant area duct.

→ Let there be no friction. Consider the control volume:

* Control volume for Rayleigh flow:



⇒ For the flow through the constant area duct by continuity equation:

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2}} \longrightarrow (1) \qquad \{\because A \text{ is constant}\}.$$

⇒ By momentum eqn:

$$P + \rho V^2 = \text{constant}.$$

$$P_1 + \rho_1 V_1^2 = P_2 + \rho_2 V_2^2$$

$$\therefore P_1 - P_2 = \rho_2 V_2^2 - \rho_1 V_1^2$$

$$= \frac{\dot{m}}{A} V_2 - \frac{\dot{m}}{A} V_1$$

$$\left[\begin{array}{l} \dot{m} = \rho A V \\ \frac{\dot{m}}{A} = \rho V \\ \therefore \rho_1 V_1 = \rho_2 V_2 = \frac{\dot{m}}{A} \end{array} \right.$$

$$\therefore \boxed{P_1 - P_2 = \frac{\dot{m}}{A}\left[V_2 - V_1\right]} \longrightarrow (2).$$

We know that $\frac{\dot{m}}{A} = \rho V = \rho_1 V_1 = \rho_2 V_2$.

$$\therefore P_1 - P_2 = \rho V\left[V_2 - V_1\right].$$

$$\therefore \boxed{P_1 - P_2 = \rho V_2^2 - \rho V_1^2} \longrightarrow (3)$$

We know that:

$$\boxed{\rho V^2 = \gamma P M^2} \text{ (Refer Fanno flow).}$$

Sub in eqn (3):

$$\therefore \boxed{P_1 - P_2 = \gamma P_2 M_2^2 - \gamma P_1 M_1^2} \longrightarrow (4)$$

Divide the eqn (4) by P_1:

$$1 - \frac{P_2}{P_1} = \frac{\gamma P_2 M_2^2}{P_1} - \gamma M_1^2$$

$$1 + \gamma M_1^2 = \frac{P_2}{P_1}\left[\gamma M_2^2 + 1\right]$$

$$\therefore \boxed{\frac{P_2}{P_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2}} \rightarrow (5)$$

⇒ By the eqn of state:

$$P = \rho R T$$

$$\therefore \boxed{\frac{P_2}{P_1} = \frac{\rho_2}{\rho_1} \times \frac{T_2}{T_1}} \rightarrow (6)$$

⇒ The mach no ratio:

$$\boxed{\frac{M_2}{M_1} = \frac{V_2 a_1}{V_1 a_2} = \frac{V_2}{V_1}\sqrt{\frac{T_1}{T_2}}} \rightarrow (7)$$

⇒ For impulse function:

$$\frac{F_2}{F_1} = \frac{P_2 A\left[1+\gamma M_2^2\right]}{P_1 A\left[1+\gamma M_1^2\right]} \qquad \left\{ \begin{array}{l} F = PA + \rho A V^2 \\ \quad = PA\left[1+\gamma M^2\right] \end{array} \right\}$$

$$\therefore \boxed{\frac{F_2}{F_1} = \frac{P_2\left[1+\gamma M_2^2\right]}{P_1\left[1+\gamma M_1^2\right]}} \rightarrow (8)$$

Sub: Eqn (5) in Eqn (8):

$$\frac{F_2}{F_1} = \frac{1+\gamma M_1^2\left[1+\gamma M_2^2\right]}{1+\gamma M_2^2\left[1+\gamma M_1^2\right]} = 1$$

$$\therefore \boxed{\frac{F_2}{F_1} = 1} \rightarrow (9)$$

The isentropic stagnation pressure ratio is given by:

we know $\frac{P_0}{P} = \left[1 + \frac{\gamma-1}{2} M^2\right]^{\frac{\gamma}{\gamma-1}}$

$$\therefore \boxed{\frac{P_{02}}{P_{01}} = \frac{P_2}{P_1}\frac{\left[1+\frac{\gamma-1}{2}M_2^2\right]^{\gamma/\gamma-1}}{\left[1+\frac{\gamma-1}{2}M_1^2\right]^{\gamma/\gamma-1}}} \rightarrow (10)$$

⇒ The entropy change:

$$\boxed{\frac{S_2 - S_1}{C_p} = \ln \frac{T_2/T_1}{\left(P_2/P_1\right)^{\frac{\gamma-1}{\gamma}}}} \rightarrow (11)$$

The relation b/w the parameters at a different states of the process. All these changes are brought about by changes in stagnation temperature. The rate of change of stream properties along Rayleigh lines is a function of the rate of change of the stagnation temperature. From the energy relation, the stagnation temperature T_0 is:

$$T_0 = T + \frac{V^2}{2C_p} = T\left(1 + \frac{V^2}{2C_pT}\right)$$

$$T_0 = T\left(1 + \frac{V^2(\gamma-1)}{2R\gamma T}\right)$$

$$\boxed{T_0 = T\left(1 + \frac{M^2(\gamma-1)}{2}\right)} \rightarrow (12) \qquad \left[\because M^2 = \frac{V^2}{a^2},\ M^2 = \frac{V^2}{\gamma R T}\right]$$

$$\boxed{\frac{T_{02}}{T_{01}} = \frac{T_2}{T_1}\frac{\left[1+\frac{M_2^2(\gamma-1)}{2}\right]}{\left[1+\frac{M_1^2(\gamma-1)}{2}\right]}} \rightarrow (13)$$

⇒ For the process involving only heat exchange, the change in stagnation temperature is a direct measure of the amount of heat transfer. If 'Q' is the heat added to the control volume, then, by the energy equation,

$$\boxed{Q = C_p(T_2 - T_1) + \frac{V_2^2 - V_1^2}{2}}$$

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$$Q = C_p T_{02} - C_p T_{01}$$

$$\therefore \boxed{Q = C_p(T_{02} - T_{01})} \longrightarrow (14)$$

→ For to get $\frac{T_2}{T_1}$, consider eqn (6) & sub value of $\frac{\rho_2}{\rho_1}$ from eqn (1): $\frac{P_2}{P_1} = \frac{\rho_2}{\rho_1} \times \frac{T_2}{T_1}$.

$$\therefore \frac{P_2}{P_1} = \frac{V_1}{V_2} \times \frac{T_2}{T_1}$$

$$\therefore \frac{T_2}{T_1} = \frac{P_2}{P_1} \times \frac{V_2}{V_1}$$

Sub the value of $\frac{V_2}{V_1}$ from eqn (7):

$$\therefore \frac{T_2}{T_1} = \frac{P_2}{P_1} \times \frac{M_2}{M_1} \times \sqrt{\frac{T_2}{T_1}}$$

$$\sqrt{\frac{T_2}{T_1}} = \frac{P_2}{P_1} \times \frac{M_2}{M_1}$$

$$\therefore \frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^2 \times \left(\frac{M_2}{M_1}\right)^2$$

sub eqn (5):

$$\therefore \boxed{\frac{T_2}{T_1} = \frac{(1+\gamma M_1^2)^2}{(1+\gamma M_2^2)^2} \times \frac{M_2^2}{M_1^2}} \longrightarrow (15)$$

sub eqn (15) in eqn (13):

$$\boxed{\frac{T_{02}}{T_{01}} = \frac{M_2^2 (1+\gamma M_1^2)^2 \left(1 + \frac{\gamma-1}{2} M_2^2\right)}{M_1^2 (1+\gamma M_2^2)^2 \left(1 + \frac{\gamma-1}{2} M_1^2\right)}} \longrightarrow (16)$$

Eqn (15) & Eqn (16) express the static & stagnation temperature ratios b/w states 1 & 2 in terms of Mach numbers at these states. Consider in state 1 $M_1 = 1$, in state 2 $M_2 = M$. Eqn (15) becomes:

$$\therefore \frac{T_2}{T_1} \Rightarrow \boxed{\frac{T}{T^*} = \frac{(1+\gamma)^2 M^2}{(1+\gamma M^2)^2}} \longrightarrow (17)$$

Eqn (16) becomes:

$$\boxed{\frac{T_0}{T_0^*} = \frac{2(1+\gamma)M^2\left(1 + \frac{\gamma-1}{2} M^2\right)}{(1+\gamma M^2)^2}} \longrightarrow (18)$$

$$\boxed{\frac{V}{V^*} = \frac{\rho^*}{\rho} = \frac{(\gamma+1)M^2}{1+\gamma M^2}} \longrightarrow (19)$$

$$\boxed{\frac{P}{P^*} = \frac{\gamma+1}{1+\gamma M^2}} \longrightarrow (20)$$

$$\boxed{\frac{P_0}{P_0^*} = \frac{\gamma+1}{1+\gamma M^2}\left[\frac{2\left[1 + \frac{\gamma-1}{2} M^2\right]}{1+\gamma}\right]^{\frac{\gamma}{\gamma-1}}} \longrightarrow (21)$$

$$\boxed{\frac{S - S^*}{C_p} = \ln M^2 \left[\frac{\gamma+1}{1+\gamma M^2}\right]^{\frac{\gamma+1}{\gamma}}} \longrightarrow (22)$$

$$\boxed{\frac{T_{02}}{T_{01}} = \frac{(T_0/T_0^*)_{M_2}}{(T_0/T_0^*)_{M_1}}} \longrightarrow (23)$$

→ In the case of subsonic flow $M_1 < 1$, when heat is added:

- Pressure decreases, $P_2 < P_1$
- Velocity and mach number increases. ($V_2 > V_1$ & $M_2 > M_1$)
- when $M_1 < \gamma^{-1/2}$, $T \uparrow$

  $M_1 > \gamma^{-1/2}$ $T \downarrow$

→ For subsonic flow when heat is added, the T ↑ses for $M_1 < \gamma^{-1/2}$ and ↓ses for $M_1 > \gamma^{-1/2}$. This is due to the fact that the value of $\frac{T}{T^*}$ goes through a maximum at $M = \frac{1}{\sqrt{\gamma}}$.

→ Total temperature ↑ses ie, $T_{02} > T_{01}$.

$\Rightarrow$ Total pressure $\downarrow$ ses ie, $P_{02} < P_{01}$.

$\Rightarrow$ For Supersonic flow ($M_1 > 1$) when heat is added to the system:

- Pressure increases, $P_2 > P_1$
- Mach number decreases, $M_2 < M_1$
- Velocity decreases, $V_2 < V_1$
- Temperature increases, $T_2 > T_1$.
- Total temperature increases, $T_{02} > T_{01}$.
- Total pressure decreases, $P_{02} < P_{01}$.